Trace inequalities for positive operators via recent refinements and reverses of Young’s inequality

Received December 29, 2017; accepted March 22, 2018

Abstract: In this paper we obtain some trace inequalities for positive operators via recent refinements and reverses of Young’s inequality due to Kittaneh-Manasrah, Liao-Wu-Zhao, Zuo-Shi-Fujii, Tominaga and Furuichi.

Keywords: Young’s inequality, Hölder operator inequality, Operator means, Arithmetic mean-Geometric mean inequality

MSC: 47A63, 47A30, 26D15, 26D10, 15A60

1 Introduction

If \( \{ e_i \}_{i \in I} \) is an orthonormal basis of \( H \), we say that \( A \in \mathcal{B}(H) \) is trace class provided

\[
\| A \|_1 := \sum_{i \in I} |\langle A e_i, e_i \rangle| < \infty.
\]

The definition of \( \| A \|_1 \) does not depend on the choice of the orthonormal basis \( \{ e_i \}_{i \in I} \). We denote by \( \mathcal{B}_1(H) \) the set of trace class operators in \( \mathcal{B}(H) \).

The following properties are also well known:

(i) We have

\[
\| A \|_1 = \| A^* \|_1
\]

for any \( A \in \mathcal{B}_1(H) \);

(ii) \( \mathcal{B}_1(H) \) is an operator ideal in \( \mathcal{B}(H) \), i.e.

\[
\mathcal{B}(H) \mathcal{B}_1(H) \mathcal{B}(H) \subseteq \mathcal{B}_1(H);
\]

(iii) \( \mathcal{B}_1(H), \| \cdot \|_1 \) is a Banach space.

We define the trace of a trace class operator \( A \in \mathcal{B}_1(H) \) to be

\[
\text{tr} (A) := \sum_{i \in I} \langle A e_i, e_i \rangle,
\]

where \( \{ e_i \}_{i \in I} \) is an orthonormal basis of \( H \). Note that this coincides with the usual definition of the trace if \( H \) is finite-dimensional. We observe that the series (1.1) converges absolutely and it is independent from the choice of basis.
The following results collect some properties of the trace:

(i) If $A \in \mathcal{B}_1(H)$ then $A^* \in \mathcal{B}_1(H)$ and

$$\text{tr} \left( A^* \right) = \overline{\text{tr}(A)}.$$

(ii) If $A \in \mathcal{B}_1(H)$ and $T \in \mathcal{B}(H)$, then $AT, TA \in \mathcal{B}_1(H)$ and

$$\text{tr} (AT) = \text{tr} (TA) \quad \text{and} \quad |\text{tr} (AT)| \leq ||A||_1 ||T||.$$

(iii) $\text{tr} (\cdot)$ is a bounded linear functional on $\mathcal{B}_1(H)$ with $||\text{tr}|| = 1$;

(iv) $\mathcal{B}_{fn}(H)$, the space of operators of finite rank, is a dense subspace of $\mathcal{B}_1(H)$.

Now, for the finite dimensional case, it is well known that the trace functional is submultiplicative, that is, for positive semidefinite matrices $A$ and $B$ in $M_n(\mathbb{C})$,

$$0 \leq \text{tr}(AB) \leq \text{tr} (A) \text{ tr} (B).$$

Therefore

$$0 \leq \text{tr}(A^k) \leq [\text{tr}(A)]^k,$$

where $k$ is any positive integer.

In 2000, Yang [31] proved a matrix trace inequality

$$\text{tr} \left( (AB)^k \right) \leq (\text{tr} A)^k (\text{tr} B)^k, \quad (1.2)$$

where $A$ and $B$ are positive semidefinite matrices over $\mathbb{C}$ of the same order and $k$ is any positive integer.

If $(H, \langle \cdot, \cdot \rangle)$ is a separable infinite-dimensional Hilbert space then the inequality (1.2) is also valid for any positive operators $A, B \in \mathcal{B}_1(H).$ This result was obtained by L. Liu in 2007, see [20].

In 2001, Yang et al. [32] improved (1.2) as follows:

$$\text{tr} \left( (AB)^m \right) \leq \left[ \text{tr} \left( A^{2m} \right) \text{ tr} \left( B^{2m} \right) \right]^{1/2}, \quad (1.3)$$

where $A$ and $B$ are positive semidefinite matrices over $\mathbb{C}$ of the same order and $m$ is any positive integer.

Stronger results than inequalities (1.2) and (1.3) had been obtained in the last 70s by Lieb and Thirring in [19].

In [25] the authors have proved many trace inequalities for sums and products of matrices. For instance, if $A$ and $B$ are positive semidefinite matrices in $M_n(\mathbb{C})$, then

$$\text{tr} \left[ (AB)^k \right] \leq \min \left\{ ||A||^k \text{ tr} \left( B^k \right), ||B||^k \text{ tr} \left( A^k \right) \right\}$$

for any positive integer $k$. Also, if $A, B \in M_n(\mathbb{C})$ then for $r \geq 1$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have the following Young type inequality

$$\text{tr} \left( \left| AB \right|^r \right) \leq \text{tr} \left[ \left( \frac{|A|^p}{p} + \frac{|B|^q}{q} \right)^r \right]. \quad (1.4)$$

Ando [1] proved a strong form of Young’s inequality · it was shown that if $A$ and $B$ are in $M_n(\mathbb{C})$, then there is a unitary matrix $U$ such that

$$|AB| = U \left( \frac{1}{p} |A|^p + \frac{1}{q} |B|^q \right) U^*,$$

where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, which immediately gives the trace inequality

$$\text{tr} \left( |AB| \right) \leq \frac{1}{p} \text{ tr} (|A|^p) + \frac{1}{q} \text{ tr} (|B|^q).$$

This inequality can also be obtained from (1.4) by taking $r = 1.$
The following Hölder’s type inequality has been obtained by Ruskai in [23]

\[ |\text{tr} (AB)| \leq \text{tr} (|AB|) \leq \left[ \text{tr} (|A|^p) \right]^{1/p} \left[ \text{tr} (|B|^q) \right]^{1/q} \]

where \( p, q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( A, B \in \mathcal{B} (H) \) with \( |A|^p, |B|^q \in \mathcal{B}_1 (H) \).

In particular, for \( p = 2 \) we get the Schwarz inequality

\[ |\text{tr} (AB)| \leq \text{tr} (|AB|) \leq \left[ \text{tr} (|A|^2) \right]^{1/2} \left[ \text{tr} (|B|^2) \right]^{1/2} \]

with \( |A|^2, |B|^2 \in \mathcal{B}_1 (H) \).

Assume that \( A, B \) are positive invertible operators on a complex Hilbert space \( (H, \langle \cdot, \cdot \rangle) \). We use the following notation

\[ A^{\frac{p}{q}} B := A^{1/2} \left( A^{-1/2} B A^{-1/2} \right)^\nu A^{1/2}, \quad \text{(1.5)} \]

for the weighted geometric mean. When \( \nu = \frac{1}{2} \), we write \( A^\frac{1}{2} B \) for brevity.

We have the following Hölder type trace inequality for the weighted geometric mean [9]: If \( A, B \) are positive invertible operators, \( p, q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( A^p, B^q \in \mathcal{B}_1 (H) \), then \( B^q A^{\frac{1}{p}} A^p \in \mathcal{B}_1 (H) \) and

\[ \text{tr} (B^q A^{\frac{1}{p}} A^p) \leq \left[ \text{tr} (A^p) \right]^{1/p} \left[ \text{tr} (B^q) \right]^{1/q}. \]

In particular, if \( A^2, B^2 \in \mathcal{B}_1 (H) \), then \( B^2 A^2 \in \mathcal{B}_1 (H) \) and

\[ \left[ \text{tr} (B^2 A^2) \right]^\nu \leq \text{tr} (A^2) \text{tr} (B^2). \]

Also, if \( A, B \) are positive invertible operators, \( p, q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( C \in \mathcal{B}_1 (H), C \geq 0 \) then \( CA^p, CB^q, C (B^q A^{\frac{1}{p}} A^p) \in \mathcal{B}_1 (H) \) and

\[ \text{tr} \left( C (B^q A^{\frac{1}{p}} A^p) \right) \leq \left[ \text{tr} (CA^p) \right]^{1/p} \left[ \text{tr} (CB^q) \right]^{1/q}. \]

In particular, if \( C \in \mathcal{B}_1 (H) \), then \( CA^2, CB^2, C (B^2 A^2) \in \mathcal{B}_1 (H) \) and

\[ \left[ \text{tr} \left( C (B^2 A^2) \right) \right]^\nu \leq \text{tr} (CA^2) \text{tr} (CB^2). \]

Related inequalities may be found in [9] as well.

For the theory of trace functionals and their applications the reader is referred to [27].

For some classical trace inequalities see [4], [6], [22] and [33], which are continuations of the work of Bellman [2]. For related works the reader can refer to [1], [3], [4], [12], [17], [20], [21], [24] and [30].

Motivated by the above results, we establish in this paper some new trace inequalities via recent scalar Young type inequalities.

## 2 Trace Inequalities Via Kittaneh-Manasrah Results

Kittaneh and Manasrah [15], [16] provided a refinement and a reverse for Young’s inequality as follows:

\[ r \left( \sqrt{a} - \sqrt{b} \right)^2 \leq (1 - \nu) a + \nu b - a^{1-\nu} b^\nu \leq R \left( \sqrt{a} - \sqrt{b} \right)^2, \quad \text{(2.1)} \]

where \( a, b > 0, \nu \in [0, 1] \), \( r = \min \{1 - \nu, \nu\} \) and \( R = \max \{1 - \nu, \nu\} \). The case \( \nu = \frac{1}{2} \) reduces (2.1) to an identity.

We can give a simple direct proof for (2.1) as follows. Recall the following result obtained by the author in 2006 [7] that provides a refinement and a reverse for the weighted Jensen’s discrete inequality:

\[ \frac{n}{\min_{j \in \{1, 2, \ldots, n\}} \{ p_j \}} \left[ \frac{1}{n} \sum_{j=1}^{n} \Phi (x_j) - \Phi \left( \frac{1}{n} \sum_{j=1}^{n} x_j \right) \right] \]

\[ \text{(2.2)} \]
\[
\leq \frac{1}{p_n} \sum_{j=1}^{n} p_j \Phi(x_j) - \Phi \left( \frac{1}{p_n} \sum_{j=1}^{n} p_j x_j \right)
\]
\[
\leq n \max_{j \in \{1,2,\ldots,n\}} \{ p_j \} \left[ \frac{1}{n} \sum_{j=1}^{n} \Phi(x_j) - \Phi \left( \frac{1}{n} \sum_{j=1}^{n} x_j \right) \right],
\]

where \( \Phi : C \to \mathbb{R} \) is a convex function defined on convex subset \( C \) of the linear space \( X \), \( \{x_j\}_{j \in \{1,2,\ldots,n\}} \) are vectors in \( C \) and \( \{p_j\}_{j \in \{1,2,\ldots,n\}} \) are nonnegative numbers with \( p_0 = \sum_{j=1}^{n} p_j > 0 \). For \( n = 2 \), we deduce from (2.2) that

\[
2 \min \{ \nu, 1 - \nu \} \left[ \frac{\Phi(x) + \Phi(y)}{2} - \Phi \left( \frac{x + y}{2} \right) \right] \leq 2 \max \{ \nu, 1 - \nu \} \left[ \frac{\Phi(x) + \Phi(y)}{2} - \Phi \left( \frac{x + y}{2} \right) \right]
\]

for any \( x, y \in \mathbb{R} \) and \( \nu \in [0,1] \). If we take \( \Phi(x) = \exp(x) \), then we get from (2.3)

\[
2 \min \{ \nu, 1 - \nu \} \left[ \exp(x) + \exp(y) - \exp \left( \frac{x + y}{2} \right) \right] \leq 2 \max \{ \nu, 1 - \nu \} \left[ \exp(x) + \exp(y) - \exp \left( \frac{x + y}{2} \right) \right]
\]

for any \( x, y \in \mathbb{R} \) and \( \nu \in [0,1] \). Further, denote \( \exp(x) = a \), \( \exp(y) = b \) with \( a, b > 0 \), then from (2.4) we obtain the inequality (2.1).

We have:

**Theorem 1.** Let \( A, B \) be two positive operators and \( P, Q \in \mathcal{B}_1(H) \) with \( P, Q > 0 \). Then for any \( \nu \in [0,1] \) we have

\[
\rho \left( \frac{\text{tr}(PA)}{\text{tr}(P)} - 2 \frac{\text{tr}(PA^{1/2}) \text{tr}(QB^{1/2})}{\text{tr}(Q)} \right) \leq (1 - \nu) \frac{\text{tr}(PA)}{\text{tr}(P)} + \nu \frac{\text{tr}(QB)}{\text{tr}(Q)} - \frac{\text{tr}(PA^{1-\nu}) \text{tr}(QB^\nu)}{\text{tr}(P) \text{tr}(Q)}
\]

\[
\leq R \left( \frac{\text{tr}(PA)}{\text{tr}(P)} - 2 \frac{\text{tr}(PA^{1/2}) \text{tr}(QB^{1/2})}{\text{tr}(Q)} \right) + \frac{\text{tr}(QB)}{\text{tr}(Q)}
\]

where \( \rho = \min \{ 1 - \nu, \nu \} \) and \( R = \max \{ 1 - \nu, \nu \} \).

**Proof.** Fix \( b > 0 \), and by using the functional calculus for the operator \( A \), we have from (2.1) that

\[
\rho \left( \langle Ax, x \rangle - 2 \sqrt{b} \langle A^{1/2} x, x \rangle + b \langle x, x \rangle \right) \leq (1 - \nu) \langle Ax, x \rangle + \nu b \langle x, x \rangle - b^\nu \langle A^{1-\nu} x, x \rangle
\]

\[
\leq R \left( \langle Ax, x \rangle - 2 \sqrt{b} \langle A^{1/2} x, x \rangle + b \langle x, x \rangle \right)
\]

for any \( x \in H \).

Now, fix \( x \in H \setminus \{0\} \). Then by using the functional calculus for the operator \( B \), we have by (2.6) that

\[
\rho \left( \langle Ax, x \rangle \|x\|^2 - 2 \langle A^{1/2} x, x \rangle \langle B^{1/2} y, y \rangle + \|x\|^2 \langle By, y \rangle \right)
\]

\[
\leq (1 - \nu) \langle Ax, x \rangle \|y\|^2 + \nu \|x\|^2 \langle By, y \rangle - \langle B^\nu y, y \rangle \langle A^{1-\nu} x, x \rangle
\]

\[
\leq R \left( \langle Ax, x \rangle \|x\|^2 - 2 \langle A^{1/2} x, x \rangle \langle B^{1/2} y, y \rangle + \|x\|^2 \langle By, y \rangle \right)
\]

for any \( x, y \in H \) and \( \nu \in [0,1] \).

This inequality is of interest in itself as well.
Now, let \(x = P^{1/2} e, y = Q^{1/2} f\) where \(e, f \in H\). Then by (2.7) we get
\[
\begin{align*}
 r \left( \left\langle P^{1/2} A P^{1/2} e, e \right\rangle \sum_{i \in I} \left( Qf_i, f_i \right) - 2 \left\langle P^{1/2} A P^{1/2} e, e \right\rangle \sum_{j \in J} \left\langle Q^{1/2} B^{1/2} Q^{1/2} f_j, f_j \right\rangle \right) \\
+ \sum_{i \in I} \left( Pe_i, e_i \right) \sum_{j \in J} \left\langle Q^{1/2} B^{1/2} Q^{1/2} f_j, f_j \right\rangle \\
\leq (1 - \nu) \sum_{i \in I} \left\langle P^{1/2} A P^{1/2} e_i, e_i \right\rangle \sum_{j \in J} \left( Qf_j, f_j \right) + \nu \sum_{i \in I} \left( Pe_i, e_i \right) \sum_{j \in J} \left\langle Q^{1/2} B^{1/2} Q^{1/2} f_j, f_j \right\rangle \\
- \sum_{i \in I} \left\langle P^{1/2} A^{1-\nu} P^{1/2} e_i, e_i \right\rangle \sum_{j \in J} \left\langle Q^{1/2} B^{\nu} Q^{1/2} f_j, f_j \right\rangle \\
\leq R \left( \sum_{i \in I} \left\langle P^{1/2} A P^{1/2} e_i, e_i \right\rangle \sum_{j \in J} \left( Qf_j, f_j \right) - 2 \sum_{i \in I} \left\langle P^{1/2} A P^{1/2} e_i, e_i \right\rangle \sum_{j \in J} \left\langle Q^{1/2} B^{1/2} Q^{1/2} f_j, f_j \right\rangle \right) \\
+ \sum_{i \in I} \left( Pe_i, e_i \right) \sum_{j \in J} \left\langle Q^{1/2} B^{1/2} Q^{1/2} f_j, f_j \right\rangle.
\end{align*}
\]
for any \(e, f \in H\).

Let \(\{e_i\}_{i \in I}\) and \(\{f_j\}_{j \in J}\) be two orthonormal bases of \(H\). If we take in (2.8) \(e = e_i, i \in I\) and \(f = f_j, j \in J\) and summing over \(i \in I\) and \(j \in J\), then we get
\[
\begin{align*}
2r \left( \sum_{i \in I} \left\langle P^{1/2} A P^{1/2} e_i, e_i \right\rangle \sum_{j \in J} \left( Qf_j, f_j \right) - 2 \sum_{i \in I} \left\langle P^{1/2} A P^{1/2} e_i, e_i \right\rangle \sum_{j \in J} \left\langle Q^{1/2} B^{1/2} Q^{1/2} f_j, f_j \right\rangle \right) \\
+ \sum_{i \in I} \left( Pe_i, e_i \right) \sum_{j \in J} \left\langle Q^{1/2} B^{1/2} Q^{1/2} f_j, f_j \right\rangle \\
\leq (1 - \nu) \sum_{i \in I} \left\langle P^{1/2} A P^{1/2} e_i, e_i \right\rangle \sum_{j \in J} \left( Qf_j, f_j \right) + \nu \sum_{i \in I} \left( Pe_i, e_i \right) \sum_{j \in J} \left\langle Q^{1/2} B^{1/2} Q^{1/2} f_j, f_j \right\rangle \\
- \sum_{i \in I} \left\langle P^{1/2} A^{1-\nu} P^{1/2} e_i, e_i \right\rangle \sum_{j \in J} \left\langle Q^{1/2} B^{\nu} Q^{1/2} f_j, f_j \right\rangle \\
\leq R \left( \sum_{i \in I} \left\langle P^{1/2} A P^{1/2} e_i, e_i \right\rangle \sum_{j \in J} \left( Qf_j, f_j \right) - 2 \sum_{i \in I} \left\langle P^{1/2} A P^{1/2} e_i, e_i \right\rangle \sum_{j \in J} \left\langle Q^{1/2} B^{1/2} Q^{1/2} f_j, f_j \right\rangle \right) \\
+ \sum_{i \in I} \left( Pe_i, e_i \right) \sum_{j \in J} \left\langle Q^{1/2} B^{1/2} Q^{1/2} f_j, f_j \right\rangle.
\end{align*}
\]
Using the properties of the trace we get
\[
\begin{align*}
&2r \left( \text{tr} \left( PA \right) \text{tr} \left( Q \right) - 2 \left( \frac{\text{tr} \left( PA^{1/2} \right)}{\text{tr} \left( P \right)} \right)^2 \right) \\
&\leq \left( 1 - \nu \right) \text{tr} \left( PA \right) \text{tr} \left( Q \right) + \nu \left( \frac{\text{tr} \left( PA^{1-\nu} \right)}{\text{tr} \left( P \right)} \right) \text{tr} \left( QB^{\nu} \right) \\
&\leq R \left( \text{tr} \left( PA \right) \text{tr} \left( Q \right) - 2 \left( \frac{\text{tr} \left( PA^{1/2} \right)}{\text{tr} \left( P \right)} \right)^2 \right),
\end{align*}
\]
and the inequality (2.5) is proved. \(\square\)

**Corollary 1.** Let \(A\) be a positive operator and \(P \in \mathcal{B}_1 \left( H \right) \) with \(P > 0\). Then for any \(\nu \in [0, 1]\) we have
\[
2r \left( \frac{\text{tr} \left( PA \right)}{\text{tr} \left( P \right)} - \left( \frac{\text{tr} \left( PA^{1/2} \right)}{\text{tr} \left( P \right)} \right)^2 \right) \leq \frac{\text{tr} \left( PA \right)}{\text{tr} \left( P \right)} - \frac{\text{tr} \left( PA^{1-\nu} \right)}{\text{tr} \left( P \right)} \frac{\text{tr} \left( PA^{\nu} \right)}{\text{tr} \left( P \right)} \leq 2r \left( \frac{\text{tr} \left( PA \right)}{\text{tr} \left( P \right)} - \left( \frac{\text{tr} \left( PA^{1/2} \right)}{\text{tr} \left( P \right)} \right)^2 \right),
\]
where \(r = \min \{ 1 - \nu, \nu \} \) and \(R = \max \{ 1 - \nu, \nu \}\).

**Remark 1.** If \(P, Q\) are positive invertible operators with \(P, Q \in \mathcal{B}_1 \left( H \right) \), then by (2.9) for \(A = P^{-1/2} Q P^{-1/2} \) we get
\[
2r \left( \frac{\text{tr} \left( Q \right)}{\text{tr} \left( P \right)} - \left( \frac{\text{tr} \left( P^{1/2} Q \right)}{\text{tr} \left( P \right)} \right)^2 \right) \leq \frac{\text{tr} \left( Q \right)}{\text{tr} \left( P \right)} - \frac{\text{tr} \left( P^{1-\nu} Q \right)}{\text{tr} \left( P \right)} \frac{\text{tr} \left( P^{\nu} Q \right)}{\text{tr} \left( P \right)} \leq 2r \left( \frac{\text{tr} \left( Q \right)}{\text{tr} \left( P \right)} - \left( \frac{\text{tr} \left( P^{1/2} Q \right)}{\text{tr} \left( P \right)} \right)^2 \right),
\]
where the operator weighted geometric mean is defined in (1.5).

**Corollary 2.** Let $A, B$ two positive operators and $P, Q \in \mathcal{B}_1(H)$ with $P, Q > 0$. If $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then we have

\[
\frac{\text{tr} (PA^p)}{\text{tr} (P)} - 2 \frac{\text{tr} (PA^{p/2}Q^{q/2})}{\text{tr} (P)} \frac{\text{tr} (Q)}{\text{tr} (Q)} + \frac{\text{tr} (QB^q)}{\text{tr} (Q)} \leq \frac{1}{p} \frac{\text{tr} (PA^p)}{\text{tr} (P)} - \frac{1}{q} \frac{\text{tr} (QB^q)}{\text{tr} (Q)} - \frac{\text{tr} (PA) \text{tr} (QB)}{\text{tr} (P) \text{tr} (Q)} \leq T \left( \frac{\text{tr} (PA^p)}{\text{tr} (P)} - 2 \frac{\text{tr} (PA^{p/2}Q^{q/2})}{\text{tr} (P)} \frac{\text{tr} (Q)}{\text{tr} (Q)} + \frac{\text{tr} (QB^q)}{\text{tr} (Q)} \right),
\]

where $t = \min \left\{ \frac{1}{p}, \frac{1}{q} \right\}$ and $T = \max \left\{ \frac{1}{p}, \frac{1}{q} \right\}$.

The proof follows by (2.5) on replacing $A$ with $A^p, B$ with $B^q$ and $\nu = \frac{1}{q}$.

**Remark 2.** If $P, Q, S, V$ are positive invertible operators with $P, Q, S, V \in \mathcal{B}_1(H)$, then by (2.10) we get for $A = P^{-1/2}SP^{-1/2}$ and $B = Q^{-1/2}VQ^{-1/2}$ that

\[
\frac{\text{tr} (P_{S_1}^p S)}{\text{tr} (P)} - \frac{\text{tr} (P_{S_1}^{p/2}Q_{S_1}^{q/2})}{\text{tr} (P)} \frac{\text{tr} (Q_{S_1}))}{\text{tr} (Q)} + \frac{\text{tr} (Q_{S_1}^q)}{\text{tr} (Q)} \leq \frac{1}{p} \frac{\text{tr} (P_{S_1}^p S)}{\text{tr} (P)} + \frac{1}{q} \frac{\text{tr} (Q_{S_1}^q)}{\text{tr} (Q)} - \frac{\text{tr} (S) \text{tr} (V)}{\text{tr} (P) \text{tr} (Q)} \leq T \left( \frac{\text{tr} (P_{S_1}^p S)}{\text{tr} (P)} - 2 \frac{\text{tr} (P_{S_1}^{p/2}Q_{S_1}^{q/2})}{\text{tr} (P)} \frac{\text{tr} (Q_{S_1}))}{\text{tr} (Q)} + \frac{\text{tr} (Q_{S_1}^q)}{\text{tr} (Q)} \right),
\]

where $t = \min \left\{ \frac{1}{p}, \frac{1}{q} \right\}$ and $T = \max \left\{ \frac{1}{p}, \frac{1}{q} \right\}$.

In particular, if we take in (2.11) $S = Q$ and $V = P$, then we get

\[
\frac{\text{tr} (P_{S_1}^p Q)}{\text{tr} (P)} - \frac{\text{tr} (P_{S_1}^{p/2}Q_{S_1}^{q/2})}{\text{tr} (P)} \frac{\text{tr} (Q_{S_1}))}{\text{tr} (Q)} + \frac{\text{tr} (Q_{S_1}^q)}{\text{tr} (Q)} \leq \frac{1}{p} \frac{\text{tr} (P_{S_1}^p Q)}{\text{tr} (P)} + \frac{1}{q} \frac{\text{tr} (Q_{S_1}^q)}{\text{tr} (Q)} - 1 \leq T \left( \frac{\text{tr} (P_{S_1}^p Q)}{\text{tr} (P)} - 2 \frac{\text{tr} (P_{S_1}^{p/2}Q_{S_1}^{q/2})}{\text{tr} (P)} \frac{\text{tr} (Q_{S_1}))}{\text{tr} (Q)} + \frac{\text{tr} (Q_{S_1}^q)}{\text{tr} (Q)} \right),
\]

where $t = \min \left\{ \frac{1}{p}, \frac{1}{q} \right\}$ and $T = \max \left\{ \frac{1}{p}, \frac{1}{q} \right\}$.

### 3 Trace Inequalities Via Liao-Wu-Zhao and Zuo-Shi-Fujii Results

We consider the Kantorovich’s ratio defined by

\[
K(h) := \frac{(h + 1)^2}{4h}, \quad h > 0.
\]

The function $K$ is decreasing on $(0, 1)$ and increasing on $[1, \infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich’s ratio holds

\[
K^r \left( \frac{a}{b} \right) a^{1-r} b^r \leq (1-r) a + r b \leq K^R \left( \frac{a}{b} \right) a^{1-R} b^R,
\]

where $a, b > 0, \nu \in [0, 1], r = \min \left\{ 1 - \nu, \nu \right\}$ and $R = \max \left\{ 1 - \nu, \nu \right\}$.

The first inequality in (3.1) was obtained by Zuo et al. in [34] while the second by Liao et al. [18]. We can give a simple direct proof for (3.1) as follows.
Indeed, if we write the inequality (2.3) for the convex function \( \Phi(x) = -\ln x \), and for the positive numbers \( a \) and \( b \) we get

\[
2 \min \{ \nu, 1 - \nu \} \left[ \ln \left( \frac{a + b}{2} \right) - \frac{\ln a + \ln b}{2} \right] \leq \ln \left[ \nu b + (1 - \nu) a \right] - (1 - \nu) \ln a - \nu \ln b
\]

\[
\leq 2 \max \{ \nu, 1 - \nu \} \left[ \ln \left( \frac{a + b}{2} \right) - \frac{\ln a + \ln b}{2} \right]
\]

that is equivalent to

\[
\min \{ \nu, 1 - \nu \} \ln \left( \frac{a + b}{2\sqrt{ab}} \right)^2 \leq \ln \left[ \frac{\nu b + (1 - \nu) a}{a^{1-\nu}b^\nu} \right]
\]

\[
\leq \max \{ \nu, 1 - \nu \} \ln \left( \frac{a + b}{2\sqrt{ab}} \right)^2
\]

and to (3.1), as stated.

If \( a \in [m_1, M_1] \) and \( b \in [m_2, M_2] \) with \( 0 < m_1 < M_1, 0 < m_2 < M_2 \) then

\[
\frac{m_1}{M_2} \leq \frac{a}{b} \leq \frac{M_1}{m_2}.
\]

Denote

\[
m := \min_{(a,b) \in [m_1,M_1] \times [m_2,M_2]} K \left( \frac{a}{b} \right) \quad \text{and} \quad M := \max_{(a,b) \in [m_1,M_1] \times [m_2,M_2]} K \left( \frac{a}{b} \right).
\]

Taking into account the properties of Kantorovich’s ratio we have

\[
m := \begin{cases} 
K \left( \frac{m_1}{M_2} \right) > 1 \text{ if } \frac{m_1}{M_2} < 1, \\
1 \text{ if } \frac{m_1}{M_2} \leq 1 \leq \frac{M_1}{m_2}, \\
K \left( \frac{m_1}{M_2} \right) > 1 \text{ if } 1 < \frac{M_1}{m_2},
\end{cases}
\]

\[
M := \max \left\{ K \left( \frac{m_1}{M_2} \right), K \left( \frac{M_1}{m_2} \right) \right\} > 1 \text{ if } \frac{m_1}{M_2} \leq 1 \leq \frac{M_1}{m_2},
\]

\[
K \left( \frac{M_1}{m_2} \right) > 1 \text{ if } 1 < \frac{m_1}{M_2},
\]

\[
= \begin{cases} 
K \left( \frac{M_1}{m_2} \right), K \left( \frac{M_1}{m_2} \right) > 1 \text{ if } \frac{m_1}{M_2} \leq 1 \leq \frac{M_1}{m_2}, \\
K \left( \frac{M_1}{m_2} \right) > 1 \text{ if } 1 < \frac{m_1}{M_2}.
\end{cases}
\]

We have the following result:

**Theorem 2.** Let \( A, B \) be two operators such that

\[
0 < m_1 I \leq A \leq M_1 I, \; 0 < m_2 I \leq B \leq M_2 I \tag{3.4}
\]

and \( P, Q \in \mathcal{B}_1(H) \) with \( P, Q > 0 \). Then for any \( \nu \in [0, 1] \), we have for \( m, M \) as defined by (3.2) and (3.3) that

\[
m \left( \frac{\text{tr} (PA^{1-\nu})}{\text{tr} (P)} \right) \left( \frac{\text{tr} (QB^{\nu})}{\text{tr} (Q)} \right) \leq (1 - \nu) \left( \frac{\text{tr} (PA)}{\text{tr} (P)} \right) + \nu \left( \frac{\text{tr} (QB)}{\text{tr} (Q)} \right) \leq M \left( \frac{\text{tr} (PA^{1-\nu})}{\text{tr} (P)} \right) \left( \frac{\text{tr} (QB^{\nu})}{\text{tr} (Q)} \right), \tag{3.5}
\]
where \( r = \min \{1 - \nu, \nu \} \) and \( R = \max \{1 - \nu, \nu \} \).

In particular, we have

\[
m^{1/2} \frac{\text{tr} \left( PA^{1/2} \right)}{\text{tr} (P)} \frac{\text{tr} \left( QB^{1/2} \right)}{\text{tr} (Q)} \leq \frac{1}{2} \left[ \frac{\text{tr} (PA)}{\text{tr} (P)} + \frac{\text{tr} (QB)}{\text{tr} (Q)} \right] \leq M^{1/2} \frac{\text{tr} \left( PA^{1/2} \right)}{\text{tr} (P)} \frac{\text{tr} \left( QB^{1/2} \right)}{\text{tr} (Q)}.
\]

Proof. From (3.1) we have

\[
m^{\nu} a^{1-\nu} b^{\nu} \leq (1 - \nu) a + \nu b \leq M^{R} a^{1-\nu} b^{\nu},
\]

where \( a \in [m_{1}, M_{1}], b \in [m_{2}, M_{2}] \) and \( \nu \in [0, 1] \).

Using the functional calculus for the operator \( A \), we have

\[
m^{\nu} b^{\nu} \left\langle A^{1-\nu} x, x \right\rangle \leq (1 - \nu) \left\langle A x, x \right\rangle + \nu \left\| x \right\|^2 \leq M^{R} b^{\nu} \left\langle A^{1-\nu} x, x \right\rangle,
\]

for any \( x \in H, b \in [m_{2}, M_{2}] \) and \( \nu \in [0, 1] \).

Using the functional calculus for \( B \) we get from (3.7) that

\[
m^{\nu} \left\langle A^{1-\nu} x, x \right\rangle \left\langle B^{\nu} y, y \right\rangle \leq (1 - \nu) \left\langle A x, x \right\rangle \left\| y \right\|^2 + \nu \left\| x \right\|^2 \left\langle B y, y \right\rangle \leq M^{R} \left\langle A^{1-\nu} x, x \right\rangle \left\langle B^{\nu} y, y \right\rangle,
\]

for any \( x, y \in H \) and \( \nu \in [0, 1] \).

This is an inequality of interest in itself as well.

Further, let \( x = P^{1/2} e, y = Q^{1/2} f \) where \( e, f \in H \). Then by (3.8) we have

\[
m^{\nu} \left\langle p^{1/2} A^{1-\nu} p^{1/2} e, e \right\rangle \left\langle Q^{1/2} B^{\nu} Q^{1/2} f, f \right\rangle \leq (1 - \nu) \left\langle p^{1/2} A p^{1/2} e, e \right\rangle \left\langle Q f, f \right\rangle + \nu \left\langle e, e \right\rangle \left\langle Q^{1/2} B Q^{1/2} f, f \right\rangle \leq M^{R} \left\langle p^{1/2} A^{1-\nu} p^{1/2} e, e \right\rangle \left\langle Q^{1/2} B^{\nu} Q^{1/2} f, f \right\rangle,
\]

for any \( e, f \in H \) and \( \nu \in [0, 1] \).

Now, on making use of a similar argument as in the proof of Theorem 1, we get the desired result (3.5).

\[\square\]

Remark 3. Let \( A, B \) be two operators such that the condition (3.4) is valid and \( P \in \mathcal{B}_{1} (H) \) with \( P > 0 \). Then for any \( \nu \in [0, 1] \), we have for \( m, M \) as defined by (3.2) and (3.3) that

\[
m^{\nu} \frac{\text{tr} \left( PA^{1-\nu} \right)}{\text{tr} (P)} \frac{\text{tr} \left( PB^{\nu} \right)}{\text{tr} (P)} \leq \frac{\text{tr} \left( P \left[ 1 - \nu \right] A + \nu B \right)}{\text{tr} (P)} \leq M^{R} \frac{\text{tr} \left( PA^{1-\nu} \right)}{\text{tr} (P)} \frac{\text{tr} \left( PB^{\nu} \right)}{\text{tr} (P)},
\]

where \( r = \min \{1 - \nu, \nu \} \) and \( R = \max \{1 - \nu, \nu \} \).

In particular, we have

\[
m^{1/2} \frac{\text{tr} \left( PA^{1/2} \right)}{\text{tr} (P)} \frac{\text{tr} \left( PB^{1/2} \right)}{\text{tr} (P)} \leq \frac{\text{tr} \left( P \left[ 1 - \nu \right] A + \nu B \right)}{\text{tr} (P)} \leq M^{1/2} \frac{\text{tr} \left( PA^{1/2} \right)}{\text{tr} (P)} \frac{\text{tr} \left( PB^{1/2} \right)}{\text{tr} (P)}.
\]

For \( 0 < m_{1} < M_{1}, 0 < m_{2} < M_{2} \) and \( p, q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \) we define

\[
m_{p, q} := \begin{cases} K \left( \frac{m_{p}}{m_{q}} \right) > 1 & \text{if} \quad \frac{m_{p}}{m_{q}} < 1, \\ 1 & \text{if} \quad \frac{m_{p}}{m_{q}} \leq \frac{m_{p}}{m_{q}}, \\ K \left( \frac{m_{p}}{m_{q}} \right) > 1 & \text{if} \quad \frac{m_{p}}{m_{q}} > 1. \end{cases}
\]

(3.9)
and
\[ M_{p,q} := \begin{cases} 
K \left( \frac{M^T_p}{m^T} \right) > 1 \text{ if } \frac{M^T_p}{m^T} < 1, \\
\max \left\{ K \left( \frac{M^T_p}{m^T} \right), K \left( \frac{M^T_q}{m^T} \right) \right\} > 1 \text{ if } \frac{M^T_p}{m^T} \leq 1 \leq \frac{M^T_q}{m^T}, \\
K \left( \frac{M^T_q}{m^T} \right) > 1 \text{ if } 1 < \frac{M^T_q}{m^T}.
\end{cases} \tag{3.10}
\]

**Corollary 3.** Let \( A, B \) be two operators such that (3.4) is valid and \( P, Q \in \mathcal{B}_1(H) \) with \( P, Q > 0 \). Then for any \( p, q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \) we have for \( m_{p,q}, M_{p,q} \) as defined by (3.9) and (3.10) that

\[ m_{p,q}^t \frac{\text{tr}(PA)}{\text{tr}(P)} \frac{\text{tr}(QB)}{\text{tr}(Q)} \leq \frac{1}{p} \frac{\text{tr}(PA)}{\text{tr}(P)} + \frac{1}{q} \frac{\text{tr}(QB)}{\text{tr}(Q)} \leq M_{p,q}^T \frac{\text{tr}(PA)}{\text{tr}(P)} \frac{\text{tr}(QB)}{\text{tr}(Q)}, \]

where \( t = \min \left\{ \frac{1}{p}, \frac{1}{q} \right\} \) and \( T = \max \left\{ \frac{1}{p}, \frac{1}{q} \right\} \).

**Proof.** From (3.4) we have

\[ 0 < m_{q}^p I \leq A^p < M_{q}^p I, \quad 0 < m_{q}^q I \leq B^q \leq M_{q}^q I. \]

By replacing \( A \) by \( A^p \), \( B \) by \( B^q \) and \( \nu = \frac{1}{q} \) in (3.5) then we get the desired result (3.11). \( \square \)

**Remark 4.** If we take \( Q = P \) in (3.11), then we get

\[ m_{p,q}^t \frac{\text{tr}(PA)}{\text{tr}(P)} \frac{\text{tr}(PB)}{\text{tr}(P)} \leq \frac{1}{p} \frac{\text{tr}(PA)}{\text{tr}(P)} + \frac{1}{q} \frac{\text{tr}(PB)}{\text{tr}(P)} \leq M_{p,q}^T \frac{\text{tr}(PA)}{\text{tr}(P)} \frac{\text{tr}(PB)}{\text{tr}(P)}. \]

For \( p = q = 2 \) we consider

\[ m_{2}^t \frac{\text{tr}(PA)}{\text{tr}(P)} \frac{\text{tr}(PB)}{\text{tr}(P)} \leq \frac{1}{p} \frac{\text{tr}(PA)}{\text{tr}(P)} + \frac{1}{q} \frac{\text{tr}(PB)}{\text{tr}(P)} \leq M_{2}^T \frac{\text{tr}(PA)}{\text{tr}(P)} \frac{\text{tr}(PB)}{\text{tr}(P)}. \]

\[ \hat{m}_2 := \begin{cases} 
K \left( \frac{M_1}{m^2} \right)^2 > 1 \text{ if } \frac{M_1}{m^2} < 1, \\
1 \text{ if } \frac{m_1}{m^2} \leq 1 \leq \frac{M_1}{m^2}, \\
K \left( \frac{M_2}{m^2} \right)^2 > 1 \text{ if } 1 < \frac{m_2}{m^2}.
\end{cases} \tag{3.12}
\]

and

\[ \hat{M}_2 := \begin{cases} 
K \left( \frac{M_1}{m^2} \right)^2 > 1 \text{ if } \frac{M_1}{m^2} < 1, \\
\max \left\{ K \left( \frac{M_1}{m^2} \right)^2, K \left( \frac{M_2}{m^2} \right)^2 \right\} > 1 \text{ if } \frac{m_1}{m^2} \leq 1 \leq \frac{M_1}{m^2}, \\
K \left( \frac{M_2}{m^2} \right)^2 > 1 \text{ if } 1 < \frac{m_2}{m^2}.
\end{cases} \tag{3.13}
\]

**Corollary 4.** Let \( A, B \) be two operators such that (3.4) is valid and \( P, Q \in \mathcal{B}_1(H) \) with \( P, Q > 0 \). Then for \( \hat{m}_2, \hat{M}_2 \) as defined by (3.12) and (3.13) we have that

\[ \hat{m}_2^{1/2} \frac{\text{tr}(PA)}{\text{tr}(P)} \frac{\text{tr}(QB)}{\text{tr}(Q)} \leq \frac{1}{p} \frac{\text{tr}(PA)}{\text{tr}(P)} + \frac{1}{q} \frac{\text{tr}(QB)}{\text{tr}(Q)} \leq \hat{M}_2^{1/2} \frac{\text{tr}(PA)}{\text{tr}(P)} \frac{\text{tr}(QB)}{\text{tr}(Q)}. \]
In particular,
\[ \tilde{m}^{1/2} \frac{\text{tr}(PA) \text{tr}(PB)}{\text{tr}(P)} \leq \frac{\text{tr} \left[ P \left( \frac{A^* + B^*}{2} \right) \right]}{\text{tr}(P)} \leq \bar{m}^{1/2} \frac{\text{tr}(PA) \text{tr}(PB)}{\text{tr}(P)}. \]

**Corollary 5.** If \( P, Q, S, V \) are positive invertible operators with \( P, Q, S, V \in \mathcal{B}_1(H) \) and for \( 0 < m_1 < M_1, 0 < m_2 < M_2, \)
\[ 0 < m_1 P \leq S \leq M_1 P, \; 0 < m_2 Q \leq V \leq M_2 Q. \] (3.14)

Then for any \( \nu \in [0, 1], \) we have for \( m, R \) as defined by (3.2) and (3.3) that
\[ m^r \frac{\text{tr}(P S^2 S P^{-1/2}) \text{tr}(Q S V)}{\text{tr}(P) \text{tr}(Q)} \leq (1 - \nu) \frac{\text{tr}(S) \text{tr}(V)}{\text{tr}(P) \text{tr}(Q)} + \nu \frac{\text{tr}(V)}{\text{tr}(Q)}, \] (3.15)

where \( r = \min \{1 - \nu, \nu\} \) and \( R = \max \{1 - \nu, \nu\}. \)

**Proof.** From (3.14) we have
\[ 0 < m_1 \leq P^{-1/2} S P^{-1/2} \leq M_1, \; 0 < m_2 \leq Q^{-1/2} V Q^{-1/2} \leq M_2. \]

If we use the inequality (3.5) for \( A = P^{-1/2} S P^{-1/2} \) and \( B = Q^{-1/2} V Q^{-1/2} \) then
\[ m^r \frac{\text{tr} \left( P \left( P^{-1/2} S P^{-1/2} \right)^{1-\nu} \right) \text{tr} \left( Q \left( Q^{-1/2} V Q^{-1/2} \right)^{1-\nu} \right)}{\text{tr}(P) \text{tr}(Q)} \leq (1 - \nu) \frac{\text{tr} \left( P P^{-1/2} S P^{-1/2} \right)}{\text{tr}(P)} + \nu \frac{\text{tr} \left( Q Q^{-1/2} V Q^{-1/2} \right)}{\text{tr}(Q)}, \]
which, by the properties of trace, is equivalent to (3.15).

**Remark 5.** If \( P, S, V \) are positive invertible operators with \( P, S, V \in \mathcal{B}_1(H) \) and for \( 0 < m_1 < M_1, 0 < m_2 < M_2, \)
\[ 0 < m_1 P \leq S \leq M_1 P, \; 0 < m_2 P \leq V \leq M_2 P, \]
then for any \( \nu \in [0, 1], \) we have for \( m, R \) as defined by (3.2) and (3.3) that
\[ m^r \frac{\text{tr}(P S^2 S \nu V)}{\text{tr}(P) \text{tr}(P)} \leq \frac{\text{tr} \left( (1 - \nu) S + \nu V \right)}{\text{tr}(P)}, \]
where \( r = \min \{1 - \nu, \nu\} \) and \( R = \max \{1 - \nu, \nu\}. \)

**In particular,** we have
\[ m^{1/2} \frac{\text{tr}(P S)}{\text{tr}(P)} \leq \frac{\text{tr} \left( \frac{S + V}{2} \right)}{\text{tr}(P)} \leq M^{1/2} \frac{\text{tr}(P S)}{\text{tr}(P)}. \]
4 Trace Inequalities Via Tominaga and Furuichi Results

We recall that Specht’s ratio is defined by [28]

\[ S(h) := \begin{cases} \frac{h^{1/r}}{e^{\ln(h^{1/r})}} & \text{if } h \in (0, 1) \cup (1, \infty) \\ 1 & \text{if } h = 1. \end{cases} \]

It is well known that \( \lim_{h \to 1} S(h) = 1 \), \( S(h) = S(\frac{1}{h}) > 1 \) for \( h > 0 \), \( h \neq 1 \). The function is decreasing on \((0, 1)\) and increasing on \((1, \infty)\).

The following inequality provides a refinement and a multiplicative reverse for Young’s inequality

\[ S\left( \left( \frac{a}{b} \right)^r \right) a^{1-\nu} b^{\nu} \leq (1 - \nu) a + \nu b \leq S\left( \frac{a}{b} \right) a^{1-\nu} b^{\nu}, \quad (4.1) \]

where \( a, b > 0, \nu \in [0, 1], r = \min \{1 - \nu, \nu\} \).

The second inequality in (4.1) is due to Tominaga [29] while the first one is due to Furuichi [11].

If \( a \in [m_1, M_1] \) and \( b \in [m_2, M_2] \) with \( 0 < m_1 < M_1, 0 < m_2 < M_2 \) then

\[ \frac{m_1}{M_2} \leq \frac{a}{b} \leq \frac{M_1}{m_2}. \]

Denote, for \( r \in (0, 1) \)

\[ \tilde{m}_r := \min_{(a,b) \in [m_1,m_1] \times [m_2,M_2]} S \left( \left( \frac{a}{b} \right)^r \right) \quad \text{and} \quad \tilde{M} := \max_{(a,b) \in [m_1,M_1] \times [m_1,m_1]} S \left( \frac{a}{b} \right). \]

Taking into account the properties of Specht’s ratio we have

\[ \tilde{m}_r := \begin{cases} S\left( \left( \frac{M_1}{m_2} \right)^r \right) > 1 & \text{if } \frac{M_1}{m_2} < 1, \\ 1 & \text{if } \frac{m_1}{M_2} \leq 1 \leq \frac{M_1}{m_2}, \\ S\left( \left( \frac{M_1}{M_2} \right)^r \right) > 1 & \text{if } 1 < \frac{m_1}{M_2}, \end{cases} \quad (4.2) \]

and

\[ \tilde{M} := \max \left\{ S\left( \frac{M_1}{m_2} \right), S\left( \frac{M_1}{M_2} \right) \right\} > 1 \text{ if } \frac{m_1}{M_2} \leq 1 \leq \frac{M_1}{m_2}, \quad (4.3) \]

We have the following result:

**Theorem 3.** Let \( A, B \) be two operators such that

\[ 0 < m_1 I \leq A \leq M_1 I, \quad 0 < m_2 I \leq B \leq M_2 I \]

and \( P, Q \in \mathcal{B}_1(H) \) with \( P, Q > 0 \). Then for any \( \nu \in [0, 1], \) we have for \( \tilde{m}_r, \tilde{M} \) as defined by \((4.2)\) and \((4.3)\) that

\[ \tilde{m}_r \frac{\operatorname{tr}(PA^{1-\nu})}{\operatorname{tr}(P)} \frac{\operatorname{tr}(QB^{\nu})}{\operatorname{tr}(Q)} \leq (1 - \nu) \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} + \nu \frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)} \leq \tilde{M} \frac{\operatorname{tr}(PA^{1-\nu})}{\operatorname{tr}(P)} \frac{\operatorname{tr}(QB^{\nu})}{\operatorname{tr}(Q)}, \quad (4.4) \]
where $r = \min \{1 - \nu, \nu\}$ and $R = \max \{1 - \nu, \nu\}$.

In particular, we have

$$\frac{\text{tr} \left( PA^{1/2} \right)}{\text{tr} (P) } \frac{\text{tr} \left( QB^{1/2} \right)}{\text{tr} (Q) } \leq \frac{1}{2} \left[ \frac{\text{tr} (PA)}{\text{tr} (P) } + \frac{\text{tr} (QB)}{\text{tr} (Q) } \right]$$

$$\leq \frac{\text{tr} \left( PA^{1/2} \right)}{\text{tr} (P) } \frac{\text{tr} \left( QB^{1/2} \right)}{\text{tr} (Q) } .$$

Proof. From (3.1) we have

$$\tilde{m}_r a^{1-\nu} b^{\nu} \leq (1 - \nu) a + \nu b \leq \tilde{M} a^{1-\nu} b^{\nu} ,$$

where $a \in [m_1, M_1]$, $b \in [m_2, M_2]$ and $\nu \in [0, 1]$.

Now, on making use of a similar argument as in the proof of Theorem 2, we get the desired result (4.4).

For $0 < m_1 < M_1$, $0 < m_2 < M_2$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we define for $r \in (0, 1)$

$$\tilde{m}_{r,p,q} := \begin{cases} S \left( \frac{m_r^p}{m_1^p} \right)^r > 1 \text{ if } \frac{m_r^p}{m_1^p} < 1, \\ 1 \text{ if } \frac{m_r^p}{m_1^p} \leq 1 \leq \frac{m_r^q}{m_2^q}, \\ S \left( \frac{m_r^q}{m_2^q} \right)^r > 1 \text{ if } 1 < \frac{m_r^q}{m_2^q} \end{cases} \quad (4.5)$$

and

$$\tilde{M}_{p,q} := \begin{cases} S \left( \frac{m_r^p}{m_1^p} \right) > 1 \text{ if } \frac{m_r^p}{m_1^p} < 1, \\ \max \left\{ S \left( \frac{m_r^p}{m_1^p} \right), S \left( \frac{m_r^q}{m_2^q} \right) \right\} > 1 \text{ if } \frac{m_r^p}{m_1^p} \leq 1 \leq \frac{m_r^q}{m_2^q}, \\ S \left( \frac{m_r^q}{m_2^q} \right) > 1 \text{ if } 1 < \frac{m_r^q}{m_2^q} \end{cases} \quad (4.6)$$

Corollary 6. Let $A, B$ be two operators such that (3.4) is valid and $P, Q \in \mathcal{B}_1 (H)$ with $P, Q > 0$. Then for any $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have for $\tilde{m}_{t,p,q}$, $\tilde{M}_{p,q}$ as defined by (4.5) and (4.6) that

$$\frac{\text{tr} \left( PA \right)}{\text{tr} (P) } \frac{\text{tr} \left( QB \right)}{\text{tr} (Q) } \leq \frac{1}{p} \frac{\text{tr} \left( PA^p \right)}{\text{tr} (P) } + \frac{1}{q} \frac{\text{tr} \left( QB^q \right)}{\text{tr} (Q) }$$

$$\leq \frac{\text{tr} \left( PA \right)}{\text{tr} (P) } \frac{\text{tr} \left( QB \right)}{\text{tr} (Q) } ,$$

where $t = \min \left\{ \frac{1}{p}, \frac{1}{q} \right\}$.

The interested reader may write similar inequalities to those in the previous section, however we do not present them here.

Acknowledgement: The author would like to thank the anonymous referees for their valuable comments that have been implemented in the final version of the paper.

References


